Schrödinger operators with complex potential but real spectrum

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Abstract

Several aspects of complex-valued potentials generating a real and positive spectrum are discussed. In particular, we construct complex-valued potentials whose corresponding Schrödinger eigenvalue problem can be solved analytically. © 1998 Elsevier Science B.V.

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1. Introduction

Quantum systems characterized by non-Hermitian Hamiltonians are of interest in several areas of theoretical physics. For example, in nuclear physics [1] one studies standard Schrödinger Hamiltonians with complex-valued potentials, which in this connection are called optical or average nuclear potentials. Non-Hermitian interactions are also discussed in field theories, for example, when studying Lee–Yang zeros [2]. Even in recent studies on localization–delocalization transitions in superconductors [3] and in the theoretical description of defraction of atoms by standing light waves [4] non-Hermitian Hamiltonians are of interest.

Recently, several authors [5–8] have studied standard one-dimensional Schrödinger Hamiltonians with complex-valued potentials giving rise to a real energy

spectrum. Some of this work [6,7] has been concentrated on the numerical study of parity and time-reversal (PT) invariant Hamiltonians, and it is believed [9] that this invariance is a sufficient criterion for the reality and positivity of the spectrum. It is the aim of this note to construct via the Darboux method new complex potentials for which the corresponding eigenvalue problem can be solved exactly. These potentials are not necessarily PT invariant but still give rise to a real and positive spectrum.

In order to make this note self-contained and also for the purpose of setting up our notation we briefly review in the next section the basics of the Darboux method. Using this method we then derive a rather general class of complex potentials giving rise to the harmonic oscillator spectrum. In particular, we note that this class of potentials also contains members which are not PT invariant. In Section 4, we will discuss the Darboux approach for the class of potentials $V(x) = -(ix)^N/2$, which has recently been discussed in some detail by Bender and Boettcher [6,7]. Finally, in Section 5 we discuss the exactly solvable potential

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$V(x) = \frac{1}{2}\exp(2ix)$, which may be considered as a superposition of the Bender–Boettcher potentials.

2. The Darboux method

In this section we briefly review the method of Darboux [10], which relates the spectral properties of a pair of standard Schrödinger Hamiltonians,

$$H_{1/2} = -\frac{1}{2}\frac{d^2}{dx^2} + V_{1/2}(x), \tag{1}$$

acting on the Hilbert space $L^2(\mathbb{R})$. Let us assume that for one of these Hamiltonians, say H_1, the spectral properties are exactly known. That is, its eigenvalues E_n and corresponding eigenfunctions ϕ_n are known explicitly,

$$H_1\phi_n(x) = E_n\phi_n(x). \tag{2}$$

For simplicity we will assume that H_1 has a purely discrete positive spectrum enumerated by $n = 0, 1, 2, \ldots,$ such that $0 < E_0 < E_1 < E_2 < \ldots$ If we now postulate that there exists a linear operator A such that

$$AH_1 = H_2A, \tag{3}$$

then the functions $\psi_n := A\phi_n \neq 0$ are obviously eigenfunctions of the other Hamiltonian H_2,

$$H_2\psi_n(x) = E_n\psi_n(x), \tag{4}$$

with the same eigenvalue $E_n > 0$.

A rather general form for an intertwining operator obeying (3) is given by

$$A = \sum_{k=0}^{N} f_k(x)\frac{d^k}{dx^k}, \tag{5}$$

where the $f_k : \mathbb{R} \to \mathbb{C}$, $k = 0, 1, \ldots, N - 1$, are (at least twice) differentiable functions to be determined via the condition (3) and f_N is an arbitrary constant.

The simplest nontrivial choice is $N = 1$,[4]

$$A = -\frac{d}{dx} + f(x), \quad f : \mathbb{R} \to \mathbb{C}. \tag{6}$$

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[4] More general approaches for standard real-valued potentials can be found in Ref. [11].

Putting this as an ansatz into Eq. (3) leads to ($f' = df/dx$, etc.)

$$(V_2 - V_1 + f')\frac{d}{dx} - [(V_2 - V_1)f + V_1' - f''/2]\mathbf{1} = 0. \tag{7}$$

As d/dx and the unit operator $\mathbf{1}$ are linearly independent, their coefficients have to vanish simultaneously. That is, we have the two conditions

$$V_2 = V_1 - f', \quad (V_2 - V_1)f + V_1' - f''/2 = 0. \tag{8}$$

The first equation expresses the potential V_2 in terms of V_1 and f, and thus allows us to eliminate V_2 from the second one leading, after an additional integration, to

$$f' + f^2 - 2V_1 = -2\varepsilon = \text{const}. \tag{9}$$

Substitution $f = u'/u$ then results in the Schrödinger-like equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + V_1(x)\right)u(x) = \varepsilon u(x), \tag{10}$$

with in general complex integration constant ε. We will, however, restrict ourselves to real ε for reasons to be given below. Note that we do not require square-integrability of u. We also note that in terms of f the two potentials read

$$V_{1/2}(x) = \frac{1}{2}f^2(x) \pm \frac{1}{2}f'(x) + \varepsilon, \tag{11}$$

that is, for real f the two Hamiltonians $H_{1/2}$ are SUSY partners [12].

The above approach may be used to construct also complex-valued potentials which generate a real-valued spectrum containing that of H_1. To be more explicit, we start with a given real potential V_1, solve Eq. (10) and thus obtain a new (in general complex) potential V_2, which is given by

$$V_2(x) = \left(\frac{u'(x)}{u(x)}\right)^2 - V_1(x) + 2\varepsilon. \tag{12}$$

Note that even for real ε the above potential may be complex by choosing a complex linear combination of the two fundamental solutions of (10). This linear combination is not arbitrary because u must not have zeros on the real line in order to lead to a well-defined Hamiltonian H_2 on $L^2(\mathbb{R})$. Note that for $\varepsilon > E_0$ any

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With the help of the intertwining relation (3) it is now easy to see that the strictly positive spectrum of H_1 forms a subset of the complete spectrum of H_2 with the corresponding eigenfunctions given by

$$\psi_n(x) = C_n A \phi_n(x)$$
$$= C_n \left(-\phi'_n(x) + \frac{u'(x)}{u(x)} \phi_n(x) \right) , \qquad (13)$$

where C_n stands for a normalization constant defined by $|C_n|^{-2} = \langle A\phi_n | A\phi_n \rangle$. Note that this normalization constant vanishes if ψ_n is not square integrable, implying that the corresponding eigenvalue does not belong to the spectrum of H_2. This, however, happens only for the square-integrable solutions of (10), which we have eliminated via the condition $\varepsilon < E_0$.

Noting that from (3) follows $H_1(A^\dagger)^* = (A^\dagger)^* H_2$ where $(A^\dagger)^* = d/dx + f(x)$, we realize that the above set of eigenfunctions (13) is only complete on $L^2(\mathbb{R}) \backslash \ker(A^\dagger)^*$. Note that $(A^\dagger)^*$ is a first-order differential operator and, therefore, the dimension of $\ker(A^\dagger)^*$ may not exceed unity. In other words, H_2 may have one additional eigenvalue which is below E_0. Note that $\ker(A^\dagger)^*$ is a one-dimensional subspace of $L^2(\mathbb{R})$ iff the differential equation $(A^\dagger)^* \psi_\varepsilon = 0$ has a solution in this Hilbert space. This solution can explicitly be given,

$$\psi_\varepsilon(x) = \frac{C_\varepsilon}{u(x)} , \qquad (14)$$

and it may easily be verified that it is an eigenfunction of H_2 with eigenvalue ε, which belongs to the spectrum of H_2 iff $\psi_\varepsilon \in L^2(\mathbb{R})$.

As conclusion of this section we note that the Darboux method may be generalized such that one can construct complex potentials generating a real spectrum which is identical to that of a self-adjoint standard Schrödinger Hamiltonian H_1, spec $H_1 = \{E_0, E_1, \ldots\}$. In some cases, via an appropriate choice of the parameter ε and the solution of (10), this complex potential may have an additional real eigenvalue ε which lies below of the spectrum of H_1. In the next section we will demonstrate this approach for the case of the harmonic potential $V_1(x) = \frac{1}{2} x^2$.

3. Complex potentials generating a harmonic spectrum

In this section we will consider as an explicit example the harmonic oscillator potential $V_1(x) = \frac{1}{2} x^2$. The spectral properties of the corresponding Schrödinger Hamiltonian H_1 are well known,

$$E_n = n + \frac{1}{2} ,$$
$$\phi_n(x) = (\sqrt{\pi}\, 2^n n!)^{-1/2} H_n(x) \exp(-\frac{1}{2} x^2/2) , \qquad (15)$$

where H_n denotes the Hermite polynomial [13] of degree $n \in \{0, 1, 2, \ldots\}$. The most general solution of the Schrödinger-like equation (10) can be given in terms of confluent hypergeometric functions [13],

$$u(x) = e^{-x^2/2} [\alpha \,_1F_1(\tfrac{1}{4}(1 - 2\varepsilon), \tfrac{1}{2}, x^2)$$
$$+ \beta x \,_1F_1(\tfrac{1}{4}(3 - 2\varepsilon), \tfrac{3}{2}, x^2)] . \qquad (16)$$

As u should not have any zero on the real line, α must not vanish and, therefore, can be set equal to unity without loss of generality. Note that an overall factor in u is irrelevant for the relevant formulas (12)–(14). From the general discussion of the last section we also have the condition $\varepsilon < E_0 = 1/2$. Finally, we have to determine for which values of the remaining parameter $\beta \in \mathbb{C}$ the general solution (16) has no zeros on the real line. These conditions can be extracted [14,15] from the asymptotic behaviour

$$u(x) = \frac{\exp(\frac{1}{2} x^2)}{|x|^{1/2+\varepsilon}} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}(1 - 2\varepsilon))} \right.$$
$$\left. + \beta \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4}(3 - 2\varepsilon))} \frac{x}{|x|} + O(1/|x|) \right). \qquad (17)$$

For $\beta \in \mathbb{R}$ a strictly positive solution is only possible if

$$|\beta| < \beta_c(\varepsilon) := 2 \frac{\Gamma(\frac{3}{4} - \frac{\varepsilon}{2})}{\Gamma(\frac{1}{4} - \frac{\varepsilon}{2})} . \qquad (18)$$

Violation of this condition results in singularities in V_2. See Refs. [14,15] for details. However, for $\beta \in \mathbb{C}/\mathbb{R}$ the solution u does not have any zero. That is, the allowed values for β are given by the complex plane with the exception of the two cuts $]-\infty, -\beta_c(\varepsilon)]$ and $[\beta_c(\varepsilon), \infty[$ on the negative and positive real line,
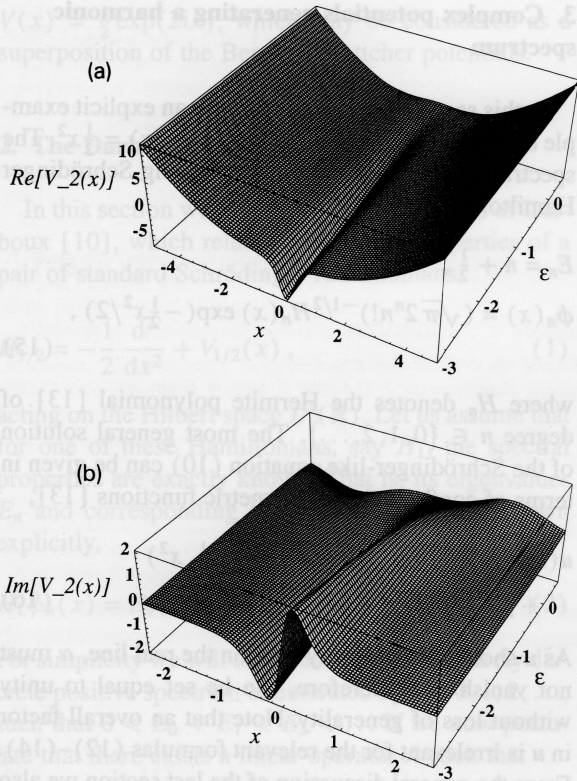
Fig. 1. (a) Real and (b) imaginary part of potential (12) with (16) for various values of ε and fixed $\beta = i$. For purely imaginary β this potential is PT invariant.

respectively. In addition, from (17) we can read off that $\psi_\varepsilon = C_\varepsilon / u$ is square integrable in the allowed ranges of the parameters. In other words, we have an additional eigenvalue $\varepsilon < 1/2$ in H_2. In Fig. 1 we show the real part (a) as well as the imaginary part (b) of the potential V_2 for $-3 < \varepsilon < 1/2$ and $\beta = i$. We note here that for all $\beta \in i\mathbb{R}$ the potential is invariant under PT transformations [6], that is, $V_2^*(-x) = V_2(x)$ and, therefore, also the Hamiltonian H_2 has this symmetry. However, other complex values of β, which are not purely imaginary, are also admissible and thus provide examples of non-PT invariant potentials generating the real harmonic oscillator spectrum with the additional ground-state energy ε.

In Figs. 2 and 3 we give several examples for fixed $\varepsilon = -1/2$ and typical values of β. Note that for this particular value of ε the solution (16) can be expressed in terms of the error function [13],
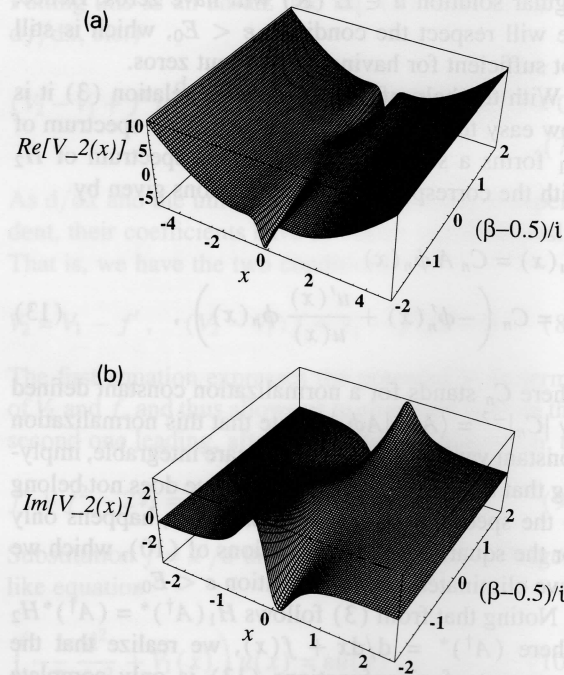

Fig. 2. Same as Fig. 1 but for fixed $\varepsilon = -1/2$ and various complex values of β with Re $\beta = 0.5$. Note that for these values of parameters the potential is not PT invariant.

$$u(x) = e^{x^2/2}\left(1 + \beta \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}(x)\right),$$

$$\beta_c(-1/2) = 2/\sqrt{\pi} \approx 1.128. \tag{19}$$

This example explicitly shows that for all

$$\beta \in \mathbb{C}\backslash\{]-\infty, -2/\sqrt{\pi}] \cup [2/\sqrt{\pi}, \infty[\} \tag{20}$$

the solution given in (19) does not have zeros on the real axis, because its imaginary part vanishes only at $x = 0$ where its real part is obviously nonzero. Fig. 2 shows (a) the real as well as (b) the imaginary part of V_2 for Re $\beta = 0.5$ and Im $\beta \in [-2, 2]$, thus crossing the real axis of the complex β-plane in the allowed region. Fig. 3 shows the same for Re $\beta = 2$ and Im $\beta \in [-1, 0]$, that is, reaching the cut of the β-plane, which is lying on the positive real axis, from below. Here a singular behaviour is clearly visible, when β approaches the cut.

We conclude this section in summarizing the spectral properties of H_2,

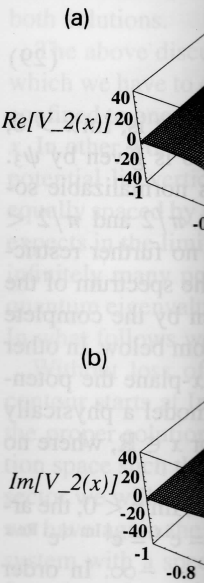$$\mathrm{spec}\,H_2 = \{\varepsilon, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots\}, \quad \psi_\varepsilon(x) = C_\varepsilon / u(x),$$

Fig. 3. Same as Fig. [...] emerges as β approach[...]

$$\psi_n(x) = \frac{1}{[\sqrt{\pi}\,2^{n+1}[...]}$$

$$\times \left[H_{n+1}(x) + [...] \right.$$

where the normal[...] termined via [5] $|C_n[...]$ $\varepsilon|\phi_n\rangle = 2(E_n - \varepsilon[...]$ of H_2 is real and b[...] of $\varepsilon < 1/2$. This [...] complex β where [...] invariant in general[...] 0, the eigenvalue p[...] $L^2(\mathbb{R})$. Let us also [...] of [14] we are abl[...] non-Hermitian Ha[...] algebra.
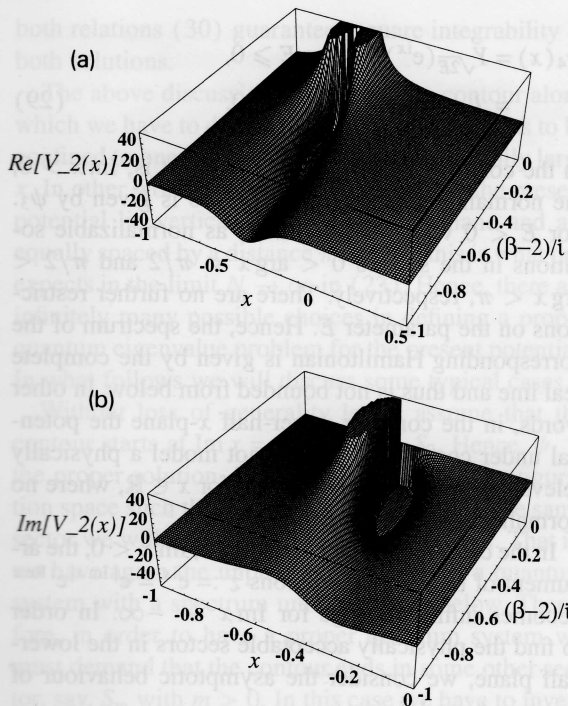
(a)

$Re[V\_2(x)]$

(b)

$Im[V\_2(x)]$

Fig. 3. Same as Fig. 2 with $\mathrm{Re}\,\beta = 2$. A singular behaviour emerges as β approaches the real axis.

$$\psi_n(x) = \frac{\exp(-\frac{1}{2}x^2)}{[\sqrt{\pi}\,2^{n+1}n!(n+1/2-\varepsilon)]^{1/2}}$$

$$\times \left[H_{n+1}(x) + \left(\frac{u'(x)}{u(x)} - x \right) H_n(x) \right], \qquad (21)$$

where the normalization constants have been determined via [5] $|C_n|^{-2} = \langle A\phi_n | A\phi_n \rangle = 2\langle \phi_n | H_1 - \varepsilon | \phi_n \rangle = 2(E_n - \varepsilon)$. We note again that the spectrum of H_2 is real and bounded below for any finite value of $\varepsilon < 1/2$. This remains true even in the case of complex β where H_2 is neither self-adjoint nor PT invariant in general. Because of $\lim_{|x|\to\infty} \mathrm{Im}\,V_2(x) = 0$, the eigenvalue problem of H_2 is well defined on $L^2(\mathbb{R})$. Let us also remark that following the approach of [14] we are able to define ladder operators for the non-Hermitian Hamiltonian H_2 closing a quadratic algebra.

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[5] Note that this is a priori only true for $\beta \in \mathbb{R}$. But as the result is independent of β we may analytically continue also to complex values.

4. Application to the Bender–Boettcher potentials

The Darboux method reviewed in Section 2 is also applicable to complex potentials. Here we will consider the class of potentials [6], $N \geqslant 2$,

$$V_1(x) := -\tfrac{1}{2}(ix)^N, \qquad (22)$$

which has recently been discussed extensively by Bender and Boettcher [6,7]. These authors have shown (numerically) that for $N \geqslant 2$ the above potential generates a real and strictly positive spectrum on an appropriate Hilbert space, which is taken to be the linear vector space of square integrable functions on some contour in the lower complex x-plane. This contour has to lie in some sectors centered about anti-Stokes lines and bounded by Stokes lines for $\mathrm{Re}\,x \to \pm\infty$ [6]. For simplicity, we will assume that this contour, which is the one-dimensional configuration space of the problem, approaches the anti-Stokes lines, that is,

$$\lim_{\mathrm{Re}\,x\to\pm\infty} \frac{x}{|x|} = \exp\left(-i\frac{\pi}{2} \pm i\frac{2\pi}{N+2} \right). \qquad (23)$$

For the present case the Schrödinger-like equation (10) cannot be solved analytically for arbitrary ε. However, for the special case $\varepsilon = 0$, which is below the ground-state energy of H_1, this equation reduces to Bessel's differential equation. In other words, for this special case we can express the solution of (10) in terms of modified Bessel functions [13],

$$u(x) = z^\nu [\alpha I_\nu(z) + \beta K_\nu(z)], \qquad (24)$$

where we have set

$$z := \frac{2}{N+2}(ix)^{(N+2)/2}, \quad \nu := \frac{1}{N+2}. \qquad (25)$$

Note that the anti-Stokes lines in the x-plane are mapped onto the negative real line in the z-plane (this is the cut of the Bessel functions) such that

$$\lim_{\mathrm{Re}\,x\to\pm\infty} \frac{z}{|z|} = e^{\pm i\pi}. \qquad (26)$$

The new potential generated via the Darboux method now reads

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[6] Note that the special case $N = 2$ corresponds to the harmonic oscillator discussed in the previous section.

$$V_2(x) = -\tfrac{1}{2}(ix)^N$$

$$\times \left[2\left(\frac{\alpha I_{\nu-1}(z) - \beta K_{\nu-1}(z)}{\alpha I_\nu(z) + \beta K_\nu(z)} \right)^2 - 1 \right], \quad (27)$$

and approaches for $|x| \to \infty$ asymptotically the original potential V_1 given in (22), that is, $V_2(x) = V_1(x)[1 + O(1/|x|)]$. As a consequence, the new potential V_2 has the same Stokes and anti-Stokes lines as the original one and, therefore, both Hamiltonians can be defined on the same Hilbert space. In other words, the Darboux transformation leaves the Hilbert space invariant.

We remark that for appropriate choices of the parameters α and β the Hamiltonian H_2 has an additional eigenvalue $\varepsilon = 0$, which is its ground-state energy. This, in fact, corresponds to a situation where SUSY is unbroken. Note that the Hamiltonians $H_{1/2}$ are SUSY partners [12]. In the case of broken SUSY H_2 has the same spectrum as H_1. By allowing for complex α and β it is also possible to construct potentials V_2 which are not PT-invariant. Nevertheless, the spectrum is identically with that of V_1 which, for $N \geqslant 2$, is strictly positive and discrete [6]. Note that in the present context PT-invariance of a potential V means $V(x) = V^*(-x)$ for all $x \in \mathbb{R}$.

5. Another exactly solvable complex potential

In this section we will consider the eigenvalue problem associated with the complex potential $V(x) = \tfrac{1}{2} \exp(2ix)$, which may be considered as a superposition of all Bender–Boettcher potentials. This potential is periodic, $V(x + \pi) = V(x)$, and, in some sense, simulates a proper regularized large N limit of (22). The corresponding eigenvalue problem can be write in the form

$$\psi''(x) + (2E - e^{2ix})\psi(x) = 0, \quad (28)$$

which in turn is reducible to Bessel's differential equation [13]. Pairwise linearly independent solutions are, for example, given by

$$\psi_1(x) = H^{(1)}_{\sqrt{2E}}(e^{ix}), \quad \psi_2(x) = H^{(2)}_{\sqrt{2E}}(e^{ix}),$$

$$\psi_3(x) = J_{\sqrt{2E}}(e^{ix}),$$

$$\psi_4(x) = Y_{\sqrt{2E}}(e^{ix}) \quad \text{for } E \geqslant 0,$$

$$= J_{-\sqrt{2E}}(e^{ix}) \quad \text{for } E < 0. \quad (29)$$

In the complex upper-half x-plane, that is, $\operatorname{Im} x > 0$, the normalizable solution for $E > 0$ is given by ψ_3. For $E < 0$ we have ψ_3 and ψ_4 as normalizable solutions in the sectors $0 < \arg x < \pi/2$ and $\pi/2 < \arg x < \pi$, respectively. There are no further restrictions on the parameter E. Hence, the spectrum of the corresponding Hamiltonian is given by the complete real line and thus is not bounded from below. In other words, in the complex upper-half x-plane the potential under consideration does not model a physically relevant system. The same holds for $x \in \mathbb{R}$, where no normalizable solutions can be found.

In the complex lower-half x-plane, $\operatorname{Im} x < 0$, the argument of the Bessel functions $z := e^{ix} = e^{|\operatorname{Im} x|}e^{i\operatorname{Re} x}$ becomes infinitely large for $\operatorname{Im} x \to -\infty$. In order to find the physically acceptable sectors in the lower-half plane, we consider the asymptotic behaviour of the pair ψ_1 and ψ_2,

$$\psi_1(x) = \sqrt{\frac{2}{\pi}}\, e^{-ix/2} \exp\left[i\left(e^{|\operatorname{Im} x|} e^{i\operatorname{Re} x} \right.\right.$$
$$\left.\left. - \frac{\pi}{2}\sqrt{2E} - \frac{\pi}{4} \right) \right][1 + O(1/z)],$$

$$\psi_2(x) = \sqrt{\frac{2}{\pi}}\, e^{-ix/2} \exp\left[-i\left(e^{|\operatorname{Im} x|} e^{i\operatorname{Re} x} \right.\right.$$
$$\left.\left. - \frac{\pi}{2}\sqrt{2E} - \frac{\pi}{4} \right) \right][1 + O(1/z)]. \quad (30)$$

The second exponential in (30) is dominating the asymptotic behaviour of ψ_1 and ψ_2. Which one of these two will be the exponentially decreasing solution depends on the sign of the imaginary part of $e^{i\operatorname{Re} x}$. As a consequence, the complex lower-half x-plane is divided into vertical stripes defined by

$$S_l = \{ x \in \mathbb{C} \mid \operatorname{Im} x < 0, l\pi \leqslant \operatorname{Re} x < (l+1)\pi \},$$
$$l \in \mathbb{Z}. \quad (31)$$

For $x \in S_{2l}$ we find that ψ_1 is the normalizable solution, whereas for $x \in S_{2l+1}$ the normalizable solution is given by ψ_2. At the borders between neighbouring stripes, that is, for $\operatorname{Re} x = l\pi$, the first exponential in

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$$E_n = \tfrac{1}{2}(n + \tfrac{1}{2})^2,$$

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have $\nu = \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{4}$
spectrum

$$E_n = \frac{1}{8l^2}(n+1)^2,$$
$$n \in \mathbb{N}_0 \setminus \{2l - 1,$$

Now we investigate
$H^{(2)}_\nu$ represents the
sponding sector. Fo

both relations (30) guarantees square integrability of both solutions.

The above discussion shows that the contour along which we have to define our quantum system has to be confined to one of these strips for asymptotically large x. In other words, the anti-Stokes lines for the present potential lie vertical in the lower-half plane and are equally spaced by a distance π. In fact, this is what one expects in the limit $N \to \infty$ in (23). Hence, there are infinitely many possible choices in defining a proper quantum eigenvalue problem for the present potential. In what follows we will discuss some typical cases.

Without loss of generality let us assume that the contour starts at $\mathrm{Im}\, x = -\infty$, say, in S_0. Hence, ψ_1 is the proper solution. If we now choose our configuration space such that this contour also ends in the same sector we will not attain any restrictions on E. That is, we have again the unphysical situation of a quantum system with a spectrum unbound from below. Therefore, in order to have a proper quantum system we must demand that the contour ends in some other sector, say, S_m with $m > 0$. In this case we have to investigate the analytic continuation of ψ_1 into this sector, which is given by [13]

$$\sin(\nu\pi) H_\nu^{(1)}(z\, e^{m\pi i}) = -\sin[(m-1)\nu\pi] H_\nu^{(1)}(z)$$
$$- e^{-\nu\pi i} \sin(m\nu\pi) H_\nu^{(2)}(z) . \qquad (32)$$

where $E = \nu^2/2$. For $m = 2l > 0$ we have to avoid an admixture of $H^{(2)}$, which is the exponentially growing solution in S_0 and S_{2l}. Hence we must demand that $\sin(2l\pi\nu) = 0$ and $\sin(\pi\nu) \neq 0$. In the particular case $l = 1$ this results in the condition $\nu = n + \frac{1}{2}$, $n = 0, 1, 2, 3, \ldots$, giving rise to the positive discrete spectrum

$$E_n = \tfrac{1}{2}(n + \tfrac{1}{2})^2 , \quad n \in \mathbb{N}_0 . \qquad (33)$$

In the general case $m = 2l$, we arrive at the conditions $2l\nu \in \mathbb{N}$ and $\nu \notin \mathbb{N}$. For example, for $l = 2$ we have $\nu = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \ldots$, leading to a "perforated" spectrum

$$E_n = \frac{1}{8l^2}(n + 1)^2 ,$$
$$n \in \mathbb{N}_0 \setminus \{2l - 1, 4l - 1, \ldots\} . \qquad (34)$$

Now we investigate the case $m = 2l + 1$ for which $H_\nu^{(2)}$ represents the normalizable solution in the corresponding sector. For the special case $l = 0$, that is, the

contour ends in the neighbouring sector S_1, we have to invoke the connection formula [13]

$$H_\nu^{(1)}(z\, e^{\pi i}) = -e^{-\nu\pi i} H_\nu^{(2)}(z) , \qquad (35)$$

which does not yield any condition on ν. In other words, the corresponding spectrum is unbound from below and thus unphysical. For $l \geq 1$ we again make use of (32) leading to $2l\nu \in \mathbb{N}$ with $\nu \notin \mathbb{N}$. These conditions are identically to those already found in the case $m = 2l$. Hence, we will obtain the same spectrum (34).

The remaining part of this section is devoted to a short semiclassical analysis of the problem (28). If we look at the potential at the left border of S_0, that is, $\mathrm{Re}\, x = 0$, the potential has the form $V(-iR) = \frac{1}{2}e^{2R}$ and thus exhibits a classical turning point for positive E. At the left border of S_m, i.e. $\mathrm{Re}\, x = m\pi$, the potential looks the same because of its periodicity. For a semiclassical analysis we need two complex classical turning points given, for example, by $x_1 = -i\ln k$ and $x_2 = m\pi - i\ln k$, $m \geq 1$. They are the solutions of $k^2 - e^{2ix} = 0$ with $k^2 = 2E$. The semiclassical quantization condition is conventionally given by

$$I = \int_{x_1}^{x_2} dx \sqrt{k^2 - e^{2ix}} = \pi(n + \tfrac{1}{2}) ,$$
$$n \in \mathbb{N}_0 . \qquad (36)$$

In order to get a real value for the integral we have to integrate along the horizontal line between x_1 and x_2 in the complex plane [6]. Taking into account the periodicity of the integrant and changing the integration variable to $z = e^{2ix}/k^2$, the integral can easily be calculated and yields $I = m\pi k$. Therefore, the semiclassical approximation leads to

$$E_n^{\mathrm{sc}} = \frac{1}{2m^2}(n + \tfrac{1}{2})^2 , \quad n \in \mathbb{N}_0 , \qquad (37)$$

which, in fact, is very similar to the exact result (34) valid for both cases $m = 2l$ and $m = 2l + 1$. However, we should note two essential failures of the semiclassical approximation. Firstly, it gives rise to a discrete spectrum even in the case $m = 1$ where the exact treatment has led to an unbound spectrum. Secondly, the semiclassical approach also fails to reproduce the perforation in the spectrum found for the cases $m \geq 2$, cf. (34).

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